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Optimal Evasion Against a Proportionally Guided Pursuer

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Introduction

PURSUIT-EVASION problems traditionally have been classified among the classical examples of differential-game theory.[1] In recent years a different approach[2-5] has been applied to these problems, namely, to fix the pursuer's strategy and to form a one-sided optimal control problem for the evader. This approach, being conceptually simpler than the former, enables more realistic models to be applied for the dynamics of the opponents. In general, the fixed pursuer's strategy has been taken as constant-gain proportional navigation, which, under some formulations, is an optimal strategy for the pursuer. The optimal control analyses have demonstrated, by applying a more realistic model for the pursuer's dynamics (such as pure time delay[3] or some finite-order system[2,4]), that the evader can guarantee a fine miss distance even in conflict with a pursuer of unlimited maneuverability. However, the formulations employed in Refs. 2–5 do not treat the evader with the same degree of accuracy as the pursuer. Evader dynamics are either ignored (assuming an ideal system) or approximated by imposing some bounds.

Our objective in this Note is to provide an analysis for the optimal evasion problem by applying equally realistic models for the pursuer and the evader, and to carry out a parametric study considering the more important parameters of these models and their influence on the optimal strategies and the payoffs. The model to be used is linear, because of the relative complexity of the problem. Consequently, the results will be valid in the vicinity of the nominal collision courses.

Mathematical Modeling

We shall make the following assumptions:

1) The pursuit-evasion conflict is two-dimensional in the horizontal plane.
2) The speeds of the pursuer (P) and the evader (E) are constant.
3) The trajectories of P and E can be linearized around their collision triangle.
4) P applies a fixed-gain proportional navigation.
5) E has complete information on P's system and on the collision course.
6) Each vehicle's acceleration is subject to a first-order lag.
7) E's lateral acceleration is bounded. (P's lateral acceleration may or may not be bounded.)

Referring to Fig. 1, by assumptions 1–3 above we get the following equations:

$$\sin(\gamma_{e_0} + \gamma_e) \doteq \sin(\gamma_{e_0}) + \cos(\gamma_{e_0})\gamma_e \tag{1}$$

$$\dot{R} = V_p \cos(\gamma_{p_0}) - V_e \cos(\gamma_{e_0}) \equiv V_p' - V_e' = \text{const} \tag{2}$$

and

$$\dot{y} = \dot{y}_e - \dot{y}_p = V_e \cos(\gamma_{e_0})\gamma_e - V_p \cos(\gamma_{p_0})\gamma_p \tag{3}$$

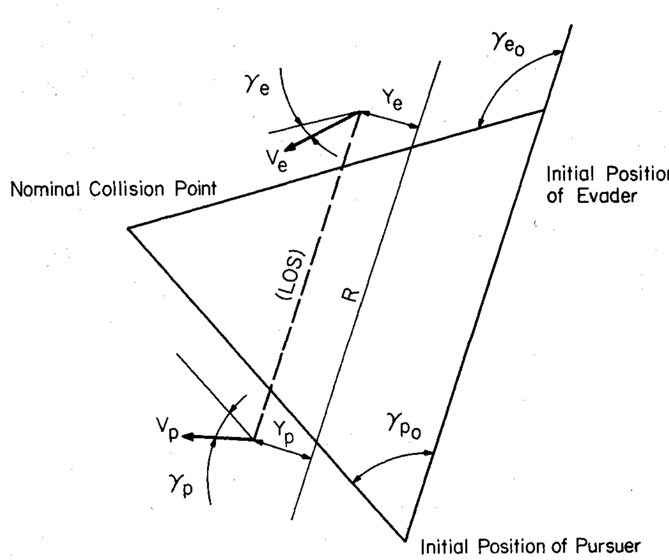

Fig. 1 Problem geometry.

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Since the nominal course leads to collision, we have the relation

$$V_e \sin(\gamma_{e_0}) - V_p \sin(\gamma_{p_0}) = 0 \tag{4}$$

P's commanded acceleration is by assumption 4

$$\ddot{y}_{p_c} = N' V_r \dot{\sigma}; \quad N' = N(V_p/V_r)\cos\gamma_{p_0} \tag{5}$$

where the line-of-sight (LOS) rate is determined by

$$\dot{\sigma} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{y}{R}\right) = \frac{y}{V_r(t_f - t)^2} + \frac{\dot{y}}{V_r(t_f - t)} \tag{6}$$

The dynamics of P are by assumption 6

$$\frac{\mathrm{d}\ddot{y}_p}{\mathrm{d}t} = \frac{\ddot{y}_{p_c} - \ddot{y}_p}{\tau_p} \tag{7}$$

where $\ddot{y}_{p_c}$ is P's commanded acceleration. A similar equation applies for E. Finally, in applying the last assumption, we shall distinguish between P whose *actual* acceleration shall be bounded by $\ddot{y}_{p_m}$ and E whose *commanded* acceleration shall be bounded (and by the sixth assumption so is its actual acceleration) by $\ddot{y}_{e_m}$. Evidently, the miss distances under this assumption are smaller than in the other possible combinations of the bounded acceleration problem.

Problem Formulation

We shall define the following state vector x:

$$x = \mathrm{col}(\dot{\gamma}_e, \gamma_e, y, \gamma_p, \dot{\gamma}_p) \tag{7}$$

and, the control function u:

$$u = (\dot{\gamma}_{e_c}/\dot{\gamma}_{e_m}) \tag{8}$$

where

$$\dot{\gamma}_{e_m} = (\ddot{\gamma}_{e_m}/V_e')$$

The linear state equations relating x to u are

$$\dot{x}_1 = -(x_1/\tau_e) + \dot{\gamma}_{e_m}(u/\tau_e) \tag{9}$$

$$\dot{x}_2 = x_1 \tag{10}$$

$$\dot{x}_3 = V_e' x_2 - V_p' x_4 \tag{11}$$

$$\dot{x}_4 = x_5 \tag{12}$$

The equation for x_5 is complicated by the limit imposed on the pursuer's turn rate.

$$\dot{x}_5 = \frac{N' V_e'}{\tau_p V_p'(t_f - t)} x_2 + \frac{N'}{\tau_p V_p'(t_f - t)^2} x_3 - \frac{N'}{\tau_p(t_f - t)} x_4 - \frac{1}{\tau_p} x_5 \quad \text{for} \quad |x_5| < \dot{\gamma}_{p_m} \tag{13a}$$

$$= 0 \quad \text{for} \quad |x_5| = \dot{\gamma}_{p_m} \tag{13b}$$

where

$$\dot{\gamma}_{p_m} = (\ddot{\gamma}_{p_m}/V_p')$$

The optimal control problem is to find u that minimizes the payoff

$$J = -x_3^2(t_f) \tag{14}$$

subject to the differential equations for $x_1(t) - x_5(t)$ with specified initial conditions $x_1(0) - x_5(0)$ and to the control bound $|u(t)| \leq 1$. The terminal time t_f is fixed and is equal to the nominal collision time.

Problem Analysis

We shall define the Hamiltonian

$$H(\lambda, x, u) = \dot{x}_1\lambda_1 + \dot{x}_2\lambda_2 + \dot{x}_3\lambda_3 + \dot{x}_4\lambda_4 + \dot{x}_5\lambda_5 \tag{15}$$

where $\dot{x}_i$ denotes the right-hand side of the associated state equation. The adjoint variables should satisfy

$$\dot{\lambda}_i = -\frac{\partial H}{\partial x_i} \tag{16}$$

The transversality conditions are as follows:

$$\lambda_i(t_f) = 0,$$

$$\lambda_3(t_f) = -2x_3(t_f) \quad \text{for} \quad i = 1,2,4,5 \tag{17}$$

And at the discontinuity point t_d, where the governing equation changes from Eq. (13a) to (13b), we get

$$\lambda_i(t_d^+) = \lambda_i(t_d^-), \quad i = 1,2,3,4 \tag{18}$$

$$\lambda_5(t_d^+) = \lambda_5(t_d^-) + c \tag{19}$$

$$H(t_d^+) = H(t_d^-) \tag{20}$$

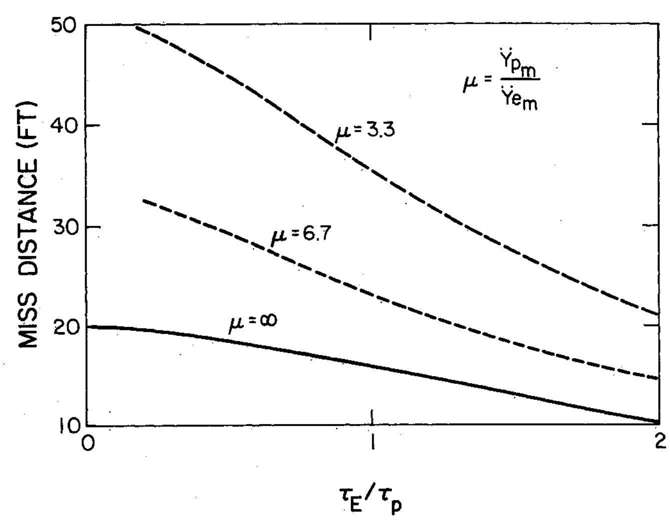

Fig. 2 Miss distance.

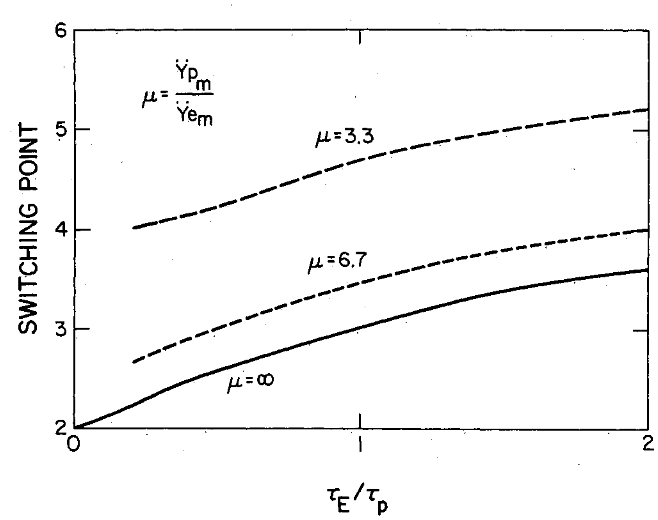

Fig. 3 Switching point.

A derivation of the maximum principle for control problems with a discontinuous system of equations is given in Ref. 6, pages 311-312. The optimality condition requires

$$u(t) = -\operatorname{sgn}\lambda_1(t) \tag{21}$$

Thus, the solution is bang-bang-type control of E's commanded acceleration similarly to the ideal system case,[4] where its actual acceleration was bang-bang.

Computational Results

The solutions for the problem were obtained numerically using a multiple shooting algorithm. The initial states are all zero and the following numerical values are used:

$$N' = 3, \qquad V_p' = 1000 \text{ fps}, \qquad V_e' = 500 \text{ fps}$$

$$\dot{\gamma}_{e_m} = 0.3 \text{ s}^{-1}, \qquad \tau_p = 0.5 \text{ s}, \qquad t_f = 3.75 \text{ s}$$

A two-parameter family of solutions has been obtained by varying $(\ddot{y}_{pm}/\ddot{y}_{em})$ and (τ_e/τ_p) (i.e., the relative maneuverability and the relative time response of the opponents are used as parameters). The miss-distance results are presented as a function of these parameters in Fig. 2. Note that E's time constant need not be smaller than P's for evading with a significant miss distance, and that a nonzero miss distance is guaranteed even when the pursuer has unlimited maneuverability.

The problem as formulated has a closed-form solution for $\tau_e = 0$, $\ddot{y}_{p_m} = \infty$, which has been obtained in Ref. 4. For this case, E applies bang-bang-type control with one switching point at $\theta \equiv (t_f - t)/\tau_p = 2$. For other values of τ_e and $\ddot{y}_{pm}$, the variable θ is shown in Fig. 3. In all of these results, the guidance parameter N' is 3.

In Ref. 4, E's system was approximated by a ramp function with t_r as the ramp time (the minimum time to change the lateral acceleration from $-\ddot{y}_{e_m}$ to $\ddot{y}_{e_m}$). Their results may be compared to the present results by approximating $t_r = 3\tau_e$. In general, the comparison is satisfactory for both the miss distance and the switching point. For a "slow" evader $(\tau_e/\tau_p > 1)$ but with relatively high maneuverability $(\ddot{y}_{em}/\ddot{y}_{pm} > 0.25)$, the ramp time approximation results are more optimistic (for E) and the predicted miss distances are greater than these of the first-order model. For a "fast" evader or for a less maneuverable one, the results are in very good agreement.

Concluding Remarks

By applying linearized kinematics to the optimal evasion problem, the optimal commanded lateral acceleration was found to be a "bang-bang" nonsingular control governed by a switching function. The optimal miss distance is dependent on the relative maneuverability of the pursuer and the evader, and on the relative time response. However, the evader need not be faster in its dynamic responses in order to guarantee a finite miss distance even in conflict with a pursuer of unlimited maneuverability.

Acknowledgment

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Analytical Solution of Optimal Trajectory-Shaping Guidance

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Introduction

RECENTLY, Lin and Tsai[1] have presented a combined mid-course and terminal guidance law design for missiles to achieve range enhancement with excellent intercept performance, and have also arrived at the closed-form solution of a closed-loop nonlinear optimum guidance law for three-dimensional flight for both the mid-course and terminal phase by neglecting some nonlinear terms.

In this Note, an analytical solution that includes terms not considered by the cited authors is presented for the preceding problem. The notation adopted would be the same as in Ref. 1.

Method of Solution

The differential Eq. (44) of Lin and Tsai[1] is used for the study and solved in analytical form. That equation is

$$\frac{dk}{dR} = \left[\left(\frac{k^2}{2} - F^2\right)\sin\sigma + \frac{k+C}{R}\right]\left(1+\sin^2\sigma\right)$$

$$R\frac{dk}{dR} = \left[\left(\frac{k^2}{2} - F^2\right)R\sin\sigma + (k+C)\right] + \left[\left(\frac{k^2}{2} - F^2\right)R\sin\sigma + (k+C)\right]\sin^2\sigma \tag{1}$$

Let $\lambda = R\sin\sigma$, then

$$\frac{d\lambda}{dR} = R\cos\sigma\frac{d\sigma}{dR} + \sin\sigma = -kR$$

by using Eq. (28) of Ref. 1.

Equation (1) leads to

$$-\frac{d^2\lambda}{dR^2} + \frac{(1+C_2)}{R}\frac{d\lambda}{dR} + F_1\lambda - C_1 = (1+\sin^2\bar{\sigma})\frac{k^2}{2}\lambda \tag{2}$$

where

$$F_1^2 = F^2 + F^2\sin^2\bar{\sigma}$$

$$C_1 = C(1+\sin^2\bar{\sigma})$$

$$C_2 = 1 + \sin^2\bar{\sigma}$$

and $\bar{\sigma}$ is the average value of σ.

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